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WHEN IS THE BOLTZMANN RELATION A 'GOOD' APPROXIMATION ?

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Frequently when treating low pressure, low temperature discharges the Boltzmann Relation ne = ne0 exp(eV/kTe) is introduced without questioning whether it is a valid assumption. The electron momentum equation in the fluid approximation is given by –

$$kT_e/mn_e\,\mathbf{grad}n_e + e/m\,\mathbf{grad}V + \mathbf{v}_e\mathbf{div.v}_e + e\mathbf{v}_e\text{x}\mathbf{B}/m + \mathbf{v}_e(Z+\nu_e) = 0 \qquad (1)$$

where m is the electron mass ne is the electron density, Te the electron temperature, $\mathbf{v}_e$ the electron velocity, **B** the applied magnetic field, Z the ionization rate by electron impact, and ν_e the electron collision frequency for momentum transfer. We label the terms in (1) as 1, 2, 3, 4, 5, and 6. The Boltzmann Relation is equivalent to (1), if and only if, terms 1 and 2 dominate over all other terms. That implies that there are no collisions (ν_e= 0), there is no volume generation (Z=0), there is no applied magnetic field (B = 0), and that electron inertia can be ignored (m = 0). Now for a collisionless plasma and when Z also is small, the third term has its maximum value at the 'plasma edge' and its relative magnitude, given that the Bohm criterion $v_i^2 = kT_e/M$, where M is the ion mass, is 'satisfied' there, is smaller by a factor m/M and so it can be justifiably ignored.

Recent work by Zimmerman *et al.* (2008), (2010) has examined the validity of the Boltzmann Relation in an active collisional plasma with an applied axial magnetic field in cylindrical geometry, thus ν_e, Z and B are all non-zero and thus terms 4, 5, and 6 cannot be ignored *a priori* and how good an approximation the Boltzmann Relation is must be examined on a case by case basis. It has also been claimed that it holds generally in a differential form at the plasma edge (Allen(2008)). Again in general this is not true as we show below.

We continue in cylindrical geometry with axisymmetry considering both ions and electrons and thus recover the results given by Forrest and Franklin (1966). Thus with unit vectors r, θ, k we introduce $n = n_e = n_i$ assuming quasi-neutrality, $u = v_{er}/c_s = v_{ir}/c_s$ as is required for the radial transport to be ambipolar, $\mathbf{B} = B_\mathbf{k}$, $\delta_e = \nu_e/Z$, $\delta_i = \nu_i/Z$, $\eta = -eV/kT_e$, $\omega_{ce} = eB/m$ and $\omega_{ci} = eB/M$ to obtain $v_{e\theta} = v_{er}\omega_{ce}/(\nu_e+Z)$, $v_{i\theta} = -v_{ir}\omega_{ci}/(\nu_e+Z)$. Introducing $R = rZ/c_s$ we end up with dimensionless equations for u, n, and η –

$$du/dR = (1 - u/R + \delta u^2)/(1 - u^2), \qquad (2)$$

$$1/n\ dn/dR = (u^2/R - u - \delta u)/(1 - u^2), \qquad (3)$$

$$\text{and, } d\eta/dR = (-u^2/R + u + \delta_i u + \delta_{em}u^3)/(1 - u^2), \qquad (4)$$

where $\delta = \delta_{em} + \delta_i$, $\delta_i = 1 + \nu_i/Z$ and $\delta_{em} = m/M\ (1 + \nu_e/Z)[1+ \omega_{ce}^2/(\nu_e + Z)^2]$, having assumed that ω_{ci} is small compared with the other frequencies, i.e. the ions are not magnetized. Near u = 1 putting u = 1-ε we find that the right hand sides of (3) and (4) behave identically with 1/ε, i.e. they go to infinity in precisely the same manner.

Now from (3) and (4) we deduce that $1/n\ dn/dR + d\eta/dR = -\ \delta_{em}$. (5)

But δ_{em} is positive definite, u is positive and monotonically increasing and (5) is not singular where u = 1, i.e. at the plasma edge, and we find that instead of (5) integrating to give $n = n_0 \exp(-\eta)$, we have – $n/n_0 = \exp(-\eta)\ .\ \exp(-\int\delta_{em}u\,dR)$ (6)

and the second term on the right hand side is a local measure of the extent to which the Boltzmann Relation is not satisfied. It is monotonically increasing in magnitude with R and thus is consistent with the results given by Zimmerman et al. who in coaxial geometry recovered (2), (3), and (4) with the origin for R and η at the point where dη/dR = 0. And it is not true that where u = 1 the Boltzmann relation is satisfied.

Plasmas in a magnetic field in this geometry were also considered by Sternberg, Godyak and Hoffman (2006) who produced a graph (Figure 12) representing the various regions for magnetized plasmas with B and p (the pressure) as axes. If one assumes that ν_e/p and ν_i/p are constant then the regions are divided by straight lines at 450 for which ω_{ce} >/< ν_e, $\omega_{ce}\omega_{ci}$>/< $\nu_e\nu_i$ and ω_{ci} >/< ν_i. They used a physically more realistic description of the charged particle motion and this results in the straight lines becoming slightly curved.

We summarize these results in a Table describing the electrons, plasma, and ions, as being magnetized M, or unmagnetized U -

	Electrons	Plasma	Ions
$\omega_{ci} > \nu_i$	M	M	M
$\omega_{ci} < \nu_i$	M	M	U
$\omega_{ce}\omega_{ci} > \nu_e\nu_i$	M	M	U
$\omega_{ce}\omega_{ci} < \nu_e\nu_i$	M	U	U
$\omega_{ce} > \nu_e$	M	U	U
$\omega_{ce} < \nu_e$	U	U	U

The transition regions coincide with $\omega_{ci} = \nu_i$, $\omega_{ci}\omega_{ce} = \nu_e\nu_i$, and $\omega_{ce} = \nu_e$, and typically range over an order of magnitude in the parameter concerned.

Zimmerman *et al.*(2008, 2010) considered the situation where the magnetic field generated by the azimuthal motion of the electrons and ions is such that it becomes comparable in magnitude to that applied. But this field varies with position since the azimuthal speeds are proportional to the radial speeds, and are greatest at the plasma edge.

They introduced a measure of departure from uniformity of the total field in a similar manner to their measure for departure from the Boltzmann Relation, in terms of an integral, but in a similar manner this effect can be given a local value varying with radial position and in their coaxial geometry is opposite in sign in the inner and outer regions. The effect is largest for $\omega_{ce} \gg \nu_e + Z$.

A full consideration of the problem of joining plasma and sheath smoothly, in the presence of a magnetic field, taking into account the rotary motion in the sheath as the electron density decreases and the ion radial motion accelerates, awaits treatment, but no doubt is simplified by the fact that the Bohm Criterion has significance, and in the sheath it is usual to assume that Z = 0 and ν_i =0, however the transition region described by Franklin and Ockendon (1970) would be modified by the need to include another parameter being the ratio of the electron gyro-radius to the plasma size .

References

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